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COMPLEX-POTENTIAL METHOD IN THE NONLINEAR THEORY OF ELASTICITY

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For materials characterized by a linear relation between Almansi strains and Cauchy stresses, relations between stresses and complex potentials are obtained and the plane static problem of the theory of elasticity is thus reduced to a boundary-value problem for the potentials. The resulting relations are nonlinear in the potentials; they generalize well-known Kolosov's formulas of linear elasticity. A condition under which the results of the linear theory of elasticity follow from the nonlinear theory considered is established. An approximate solution of the nonlinear problem for the potentials is obtained by the small-parameter method, which reduces the problem to a sequence of linear problems of the same type, in which the zeroth approximation corresponds to the problem of linear elasticity. The method is used to obtain both exact and approximate solutions for the problem of the extension of a plate with an elliptic hole. In these solutions, the behavior of stresses on the hole contour is illustrated by graphs.

1. For plane deformation of an elastic body, the strength analysis is based on solution of the plane problem. In the present paper, we consider the plane static problem for stresses within the framework of the nonlinear theory of elasticity using the Almansi and Cauchy tensors used as the strain and stress measures, respectively. The static problem of the nonlinear theory of elasticity in the actual-state variables is described by the following equations of equilibrium and continuity equations, Murnaghan's stress–strain relations, representations of strains and their invariants in terms of displacements, and boundary conditions (in particular, force conditions) on the body surface [1, 2]:

$$\operatorname{div} P + \rho \boldsymbol{f} = 0, \qquad V = \rho/\rho_0 = \sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2 - 8\varepsilon_3},$$

$$\varepsilon_1 = \operatorname{tr}\varepsilon, \qquad \varepsilon_2 = (\operatorname{tr}\varepsilon)^2 - \operatorname{tr}\varepsilon^2, \qquad \varepsilon_3 = \det\varepsilon, \qquad P = V(G - 2\varepsilon)\cdot\frac{dF}{d\varepsilon},$$

$$2\varepsilon = \nabla\boldsymbol{u} + \boldsymbol{u}\nabla - (\nabla\boldsymbol{u})\cdot(\boldsymbol{u}\nabla), \qquad P\cdot\boldsymbol{n} = \boldsymbol{p} \quad \text{on} \quad \Sigma.$$

Here $\boldsymbol{u}$, $\boldsymbol{n}$, $\boldsymbol{f}$, and $\boldsymbol{p}$ are the displacement vector, the outward normal, and the mass- and surface-force vectors, respectively, G, P, and ε are the metric tensor, the Cauchy stress tensor, and the Almansi strain tensor, respectively, ε_1, ε_2, ε_3, F, ρ, ρ_0, and V are the basic strain invariants, the density of the elastic potential, and the initial and relative densities of the material, and Σ is the surface of the deformed body. For two-dimensional deformation of an isotropic body, these relations define the following plane problem of elasticity in the complex coordinates of the actual state $z = x + iy$ and $\bar z = x - iy$ (x and y are Cartesian coordinates) [3–5]:

$$\frac{\partial P^{11}}{\partial z} + \frac{\partial P^{12}}{\partial \bar z} + \rho f^1 = 0, \quad V = \sqrt{1 - 2\varepsilon_1 + 4\varepsilon_2}, \quad \varepsilon_1 = \varepsilon^{12}, \quad 4\varepsilon_2 = (\varepsilon^{12})^2 - \varepsilon^{11}\varepsilon^{22},$$

$$P^{11} = \bar P^{22} = 2M(\varepsilon_1, \varepsilon_2)\varepsilon^{11}, \qquad P^{12} = P^{21} = 2N(\varepsilon_1, \varepsilon_2)\varepsilon^{12}, \tag{1}$$

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$$\varepsilon^{11} = \bar{\varepsilon}^{22} = 2\,\frac{\partial u^1}{\partial \bar{z}}\Big(1 - \frac{\partial u^2}{\partial \bar{z}}\Big), \qquad 1 - \varepsilon^{12} = \Big(1 - \frac{\partial u^1}{\partial z}\Big)\Big(1 - \frac{\partial u^2}{\partial \bar{z}}\Big) + \frac{\partial u^1}{\partial \bar{z}}\,\frac{\partial u^2}{\partial z},$$

$$P^{12}\,\frac{dz}{ds} - P^{11}\,\frac{d\bar{z}}{ds} = 2ip^1(s) \qquad \text{on} \quad L,$$

where M and N are the coefficients of elasticity, L is the section of the body cut by the deformation plane, and s is the arc coordinate of a contour point.

Relation (1) contains contravariant complex components of vectors and tensors (denoted by numerical superscripts), which are related to Cartesian components of the corresponding quantities by the transformation formulas [4]

$$u^1 = \bar{u}^2 = u_x + iu_y, \quad P^{11} = \bar{P}^{22} = P_{xx} - P_{yy} + 2iP_{xy}, \quad P^{12} = P_{xx} + P_{yy}; \tag{2}$$

Murnaghan's law is written in a form similar to Hooke's law, with the significant difference that the coefficients of elasticity are functions of the basic strain invariants and are expressed in terms of the elastic potential by the formulas

$$2M = -V\Big(2\,\frac{\partial F}{\partial \varepsilon_1} + \frac{\partial F}{\partial \varepsilon_2}\Big), \qquad 2\varepsilon_1 N = V\Big[2(1-\varepsilon_1)\,\frac{\partial F}{\partial \varepsilon_1} + (\varepsilon_1 - 4\varepsilon_2)\,\frac{\partial F}{\partial \varepsilon_2}\Big]. \tag{3}$$

Inversion of these formulas gives the expressions for the potential gradients:

$$V^3\,\frac{\partial F}{\partial \varepsilon_1} = (\varepsilon_1 - 4\varepsilon_2)M + \varepsilon_1 N, \qquad V^3\,\frac{\partial F}{\partial \varepsilon_2} = -2(1-\varepsilon_1)M - 2\varepsilon_1 N.$$

which lead to the following differential condition for the coefficients of elasticity:

$$2\,\frac{\partial[\varepsilon_1 N + (1-\varepsilon_1)M]}{\partial \varepsilon_1} + \frac{\partial[\varepsilon_1 N + (\varepsilon_1 - 4\varepsilon_2)M]}{\partial \varepsilon_2} + 6M = 0. \tag{4}$$

2. Eliminating the displacements and strains from system (1), we obtain equations for stresses. Elimination of the displacements and their first and second derivatives from the expressions for strains leads to the strain compatibility equation [5]

$$4[(1-\varepsilon^{12})^2 - |\varepsilon^{11}|^2]\,\mathrm{Re}\,(\varepsilon^{11}_{zz} - \varepsilon^{12}_{z\bar{z}})$$
$$= (1-\varepsilon^{12})[|\varepsilon^{11}_{\bar{z}}|^2 + |2\varepsilon^{12}_{\bar{z}} - \varepsilon^{11}_{z}|^2 - 2|\varepsilon^{11}_{z}|^2] + 2\mathrm{Re}\{\varepsilon^{11}[\varepsilon^{22}_{z}(2\varepsilon^{12}_{\bar{z}} - \varepsilon^{11}_{z}) - (\varepsilon^{22}_{\bar{z}})^2]\}. \tag{5}$$

To eliminate the strains, we first convert from ε_1 and ε_2 to the strain invariants J_1 and J_2, which are related to the complex components by the simpler expressions $\varepsilon_1 = J_1$, $4\varepsilon_2 = J_1^2 - J_2$, $J_1 = \varepsilon^{12}$, and $J_2 = \varepsilon^{11}\varepsilon^{22}$. Thus, the continuity equation and Murnaghan's law in (1), and the condition for the coefficients of elasticity (4) become

$$V = \sqrt{(1-J_1)^2 - J_2}, \quad P^{11} = \bar{P}^{22} = 2M(J_1, J_2)\varepsilon^{11}, \quad P^{12} = 2N(J_1, J_2)\varepsilon^{12},$$
$$\frac{\partial[J_1 N + (1-J_1)M]}{\partial J_1} - 2\,\frac{\partial[J_1(1-J_1)N + J_2 M]}{\partial J_2} + 3M = 0. \tag{6}$$

We introduce the basic stress invariants I_1 and I_2 having the same structure as the strain invariants J_1 and J_2: $I_1 = P^{12}$ and $I_2 = P^{11}P^{22}$. We assume that the conditions

$$\frac{\partial(J_1, J_2)}{\partial(I_1, I_2)} \neq 0, \qquad \frac{\partial(J_1, J_2)}{\partial(I_1, I_2)} \neq \infty, \tag{7}$$

which ensure reversibility of relations between the pairs of invariants J_1 and J_2 and I_1 and I_2, are satisfied. Murnaghan's law can be inverted and written as

$$\varepsilon^{11} = \bar{\varepsilon}^{22} = 2R(I_1, I_2)P^{11}, \qquad \varepsilon^{12} = 2T(I_1, I_2)P^{12}, \tag{8}$$

where R and T is another pair of the coefficients of elasticity. It follows from (6) and (8) that the coefficients of elasticity of different types and the stress and strain invariants are related by the formulas

$$4MR = 1, \quad 4NT = 1, \quad I_1 = 2NJ_1, \quad I_2 = 4M^2 J_2; \quad J_1 = 2TI_1, \quad J_2 = 4R^2 I_2. \tag{9}$$

Formulas (8) and (9) enable one to eliminate the strain components and invariants from Eqs. (5) and (6) and express them in terms of stresses. Supplementing these by the equation of equilibrium and boundary condition from (1), we arrive at the following problem for stresses:

$$\frac{\partial P^{11}}{\partial z}+\frac{\partial P^{12}}{\partial \bar z}+\rho_0 V f^1=0, \qquad V=\sqrt{(1-2TP^{12})^2-4R^2P^{11}P^{22}},$$

$$2V^2\mathrm{Re}\Big[\frac{\partial^2 RP^{11}}{\partial z^2}-\frac{\partial^2 TP^{12}}{\partial z\,\partial \bar z}\Big]=(1-2TP^{12})\Big[\Big|\frac{\partial RP^{11}}{\partial \bar z}\Big|^2+\Big|2\frac{\partial TP^{12}}{\partial \bar z}-\frac{\partial RP^{11}}{\partial z}\Big|^2-2\Big|\frac{\partial RP^{11}}{\partial z}\Big|^2\Big]$$

$$+4\mathrm{Re}\Big\{RP^{11}\Big[\frac{\partial RP^{22}}{\partial z}\Big(2\frac{\partial TP^{12}}{\partial \bar z}-\frac{\partial RP^{11}}{\partial z}\Big)-\Big(\frac{\partial RP^{22}}{\partial \bar z}\Big)^2\Big]\Big\}, \tag{10}$$

$$P^{12}\frac{dz}{ds}-P^{11}\frac{d\bar z}{ds}=2ip^1(s) \quad \text{on} \quad L,$$

$$2R^2+(1-2I_1T)\Big(2I_1\frac{\partial T}{\partial I_2}-\frac{\partial R}{\partial I_1}\Big)+4I_2(T+R)\frac{\partial R}{\partial I_2}+4I_1I_2\Big(\frac{\partial T}{\partial I_1}\frac{\partial R}{\partial I_2}-\frac{\partial T}{\partial I_2}\frac{\partial R}{\partial I_1}\Big)=0.$$

3. We consider plane deformation in the absence of mass forces for materials characterized by a linear relation between the Almansi and Cauchy tensors:

$$f^1=0, \qquad R=\text{const}, \qquad T=\text{const}. \tag{11}$$

In this case, condition (10) for the coefficients of elasticity implies that one of the coefficients can be arbitrary and the other must vanish:

$$R=0, \qquad T=\text{const}. \tag{12}$$

We note that, in this case, the strain invariants are dependent quantities: $J_1=2TI_1$ and $J_2=0$, and the condition of reversibility of the invariants (7) fails. It follows from (9) that the coefficients M and N must have the values

$$M=1/(4R)=\infty, \qquad N=1/(4T)=\text{const}. \tag{13}$$

According to (3), the coefficients (13) are related to the elastic potential $F(J_1,J_2)$ by the formulas

$$M=-V\frac{\partial F}{\partial J_1}+2(1-J_1)V\frac{\partial F}{\partial J_2}, \qquad N=\frac{1-J_1}{J_1}V\frac{\partial F}{\partial J_1}-\frac{2J_2}{J_1}V\frac{\partial F}{\partial J_2},$$

$$V=\sqrt{(1-J_1)^2-J_2}.$$

The second relation can be regarded as a differential equation for the elastic potential, whose general solution has the form

$$F(J_1,J_2)=\Theta\Big(\frac{1-J_1}{V}\Big)+\frac{1}{4T}\Big(1+\frac{J_1}{V}+\frac{1-J_1}{V}\ln V\Big), \tag{14}$$

where Θ is an arbitrary function. The potential (14) defines the class of materials for which one of the coefficients of elasticity is constant and the other is variable:

$$N=\frac{1}{4T}=\text{const}, \qquad M=\Theta'\Big(\frac{1-J_1}{V}\Big)-\frac{1-\ln V}{4T}. \tag{15}$$

The values of the coefficients (13) can be treated as the limiting values of the coefficients (15) when $J_2\to 0$ and $\big|\Theta'((1-J_1)/V)\big|=|\Theta'(1)|=\infty$. In this case, the limiting value of the potential (14) has the form $F(J_1,0)=\Theta(1)+(1/(4T))[1/(1-J_1)-\ln(1/(1-J_1))]$. In order for the potential to be finite, the quantity $\Theta(1)$ must be bounded. Thus, a material with a linear stress–strain relation belongs to the class of materials whose elastic potential has the form (14) and is characterized by the conditions $J_2\to 0$, $|\Theta|\to|\Theta(1)|<\infty$, and $|\Theta'|\to|\Theta'(1)|=\infty$.

Under conditions (11) and (12), the plane problem (10) is simplified and takes the form [6]

$$\frac{\partial P^{11}}{\partial z}+\frac{\partial P^{12}}{\partial \bar z}=0, \qquad V=1-2TP^{12}; \tag{16}$$

$$(1-2TP^{12})\frac{\partial^2 2TP^{12}}{\partial z\,\partial \bar z}+\frac{\partial 2TP^{12}}{\partial \bar z}\frac{\partial 2TP^{12}}{\partial z}=0; \tag{17}$$

$$P^{12}\frac{dz}{ds}-P^{11}\frac{d\bar z}{ds}=2ip^1 \quad \text{on} \quad L. \tag{18}$$

The equations of the problem admit full integration and representation of stresses in terms of the complex potentials, and the boundary condition leads to a boundary-value problem for the potentials. Indeed, Eq. (17) is representable as a second-order nonlinear equation for the relative density $V\,\partial^2 V/\partial z\,\partial\bar z-(\partial V/\partial \bar z)(\partial V/\partial z)=0$, whose general solution is expressed in terms of an arbitrary function [the complex potential $\sigma'(z)$] in the form $V=\sigma'(z)\bar\sigma'(\bar z)$. Thus, the density ρ and the stress P^{12} are given by

$$\rho=\rho_0\sigma'(z)\bar\sigma'(\bar z), \qquad P^{12}=(1/(2T))[1-\sigma'(z)\bar\sigma'(\bar z)]. \tag{19}$$

With allowance for (19), the equation of equilibrium (16) defines the stress P^{11}:

$$P^{11}=(1/(2T))[\sigma(z)\bar\sigma''(\bar z)-\bar\tau'(\bar z)]. \tag{20}$$

Here $\tau'(z)$ is an arbitrary function (the second complex potential). Substituting the stresses (19) and (20) into condition (18) and integrating along the contour, we obtain the following boundary-value problem for the complex potentials:

$$z-\sigma(z)\bar\sigma'(\bar z)+\bar\tau(\bar z)=4T(g^1(s)+C) \quad \text{on} \quad L, \qquad g^1(s)=i\int_0^s p^1(s)\,ds \quad (C=\text{const}). \tag{21}$$

Thus, in the variant of nonlinear elasticity considered, the stresses and density are expressed in terms of two complex potentials by nonlinear formulas and the potentials, in turn, must be determined from a nonlinear boundary condition.

The problem for stresses (16)–(18) can also be formulated for a stress function. We express the stresses in terms of the real stress function $U(z,\bar z)$ by the formulas

$$P^{11}=\bar P^{22}=-4\frac{\partial^2 U}{\partial \bar z^2}, \qquad P^{12}=4\frac{\partial^2 U}{\partial z\,\partial\bar z}. \tag{22}$$

As a result, the equation of equilibrium is satisfied identically, and the strain compatibility equation and the boundary condition constitute the following boundary-value problem for the stress function:

$$(1-2T\Delta U)\Delta\Delta U+8T(\Delta U)_{\bar z}(\Delta U)_z=0 \qquad (\Delta U=4U_{z\bar z}); \tag{23}$$

$$2U_{\bar z}=(2U_{\bar z})_0+i\int_0^s p^1(s)\,ds. \tag{24}$$

This problem can also be reduced to a problem for potentials. Integration of Eq. (23) yields the nonlinear relation between the stress function and the potentials $\sigma(z)$ and $\tau(z)$:

$$2U=\frac{1}{4T}\Big(z\bar z-\sigma(z)\bar\sigma(\bar z)+\int\tau(z)\,dz+\int\bar\tau(\bar z)\,d\bar z\Big). \tag{25}$$

With allowance for (25), formulas (22) are transformed into (19) and (20) and condition (24) becomes the boundary condition for the potentials (21).

For a more convenient comparison between the results obtained and results of the theory of linear elasticity, we convert from $\sigma(z)$ and $\tau(z)$ to the complex potentials $\varphi(z)$ and $\psi(z)$ using the formulas

$$\sigma(z)=z-4T\varphi(z), \qquad \tau(z)=4T\psi(z). \tag{26}$$

Then, the stresses (19) and (20), the stress function (25), and boundary condition (21) are given by

$$P^{11} = \bar{P}^{22} = -2[z\bar{\varphi}''(\bar{z}) + \bar{\psi}'(\bar{z}) - 4T\varphi(z)\bar{\varphi}''(\bar{z})], \quad P^{12} = 2[\varphi'(z) + \bar{\varphi}'(\bar{z}) - 4T\varphi'(z)\bar{\varphi}'(\bar{z})],$$

$$2U = \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \int \psi(z)\,dz + \int \bar{\psi}(\bar{z})\,d\bar{z} - 4T\varphi(z)\bar{\varphi}(\bar{z}), \tag{27}$$

$$\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z}) - 4T\varphi(z)\bar{\varphi}'(\bar{z}) = g^1(s) + C \quad \text{on} \quad L.$$

These formulas generalize the Kolosov formulas of linear elasticity [7, 8]:

$$P^{11} = \bar{P}^{22} = -2[z\bar{\varphi}''(\bar{z}) + \bar{\psi}'(\bar{z})], \qquad P^{12} = 2[\varphi'(z) + \bar{\varphi}'(\bar{z})],$$

$$2U = \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \int \psi(z)\,dz + \int \bar{\psi}(\bar{z})\,d\bar{z}, \quad \varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z}) = g^1(s) + C \quad \text{on} \quad L, \tag{28}$$

and differ from the latter only by terms that are nonlinear in the potentials and contain the elastic constant of the material as a factor. We now establish a condition under which (28) follow from formulas (27).

Let P_0 and L_0 be the characteristic stress and dimension and $\alpha = 4TP_0$ be the dimensionless parameter (characteristic dimensionless stress). We express the examined quantities in terms of the corresponding dimensionless quantities denoted by asterisk: $P^{11} = P_0 P_*^{11}$, $P^{12} = P_0 P_*^{12}$, $U = P_0 L_0^2 U_*$, $z = L_0 z_*$, $\varphi = P_0 L_0 \varphi_*$, $\psi = P_0 L_0 \psi_*$, $g^1 = P_0 L_0 g_*^1$, and $C = P_0 L_0 C_*$ and introduce these into (27). As a result, we have

$$P_*^{11} = \bar{P}_*^{22} = -2[z_*\bar{\varphi}_*'' + \bar{\psi}_*'] + 2\alpha\varphi_*\bar{\varphi}_*'', \qquad P_*^{12} = 2[\varphi_*' + \bar{\varphi}_*'] - 2\alpha\varphi_*'\bar{\varphi}_*',$$

$$2U_* = \bar{z}_*\varphi_* + z_*\bar{\varphi}_* + \int \psi_*\,dz_* + \int \bar{\psi}_*\,d\bar{z}_* - \alpha\varphi_*\bar{\varphi}_*, \tag{29}$$

$$\varphi_* + z_*\bar{\varphi}_*' + \bar{\psi}_* - \alpha\varphi_*\bar{\varphi}_*' = g_*^1 + C_*.$$

Assuming that all the dimensionless quantities have finite magnitudes in a closed plane domain and the dimensionless parameter is small compared to unity ($\alpha \approx 0$), in (29) we can neglect small terms (containing the parameter) compared to terms of finite magnitude. As a result, formulas (29) (after reverting to dimensional quantities) coincide with the formulas of linear elasticity (28). Thus, for elastic materials characterized by a linear stress–strain relation, the formulas of linear elasticity follow from the nonlinear formulas when the characteristic dimensionless stress is small ($\alpha \ll 1$).

4. Conformal mapping reduces the boundary-value problem for the potentials in a singly-connected (finite or infinite) plane domain S (with boundary L) to a boundary-value problem for a unit circle K (with circumference γ) or the exterior of the circle. Let the conformal mapping be specified by the function $z = w(\zeta)$, $w'(\zeta) \neq 0$, and $\zeta = r\exp(i\theta) \in K$. Then, the complex potentials take the form $\varphi(z) = \varphi(\zeta)$, $\varphi'(z) = \varphi'(\zeta)/w'(\zeta) = \Phi(\zeta)$, $\psi(z) = \psi(\zeta)$, and $\psi'(z) = \psi'(\zeta)/w'(\zeta) = \Psi(\zeta)$, and the stresses (27) become

$$P^{11} = \bar{P}^{22} = -(2/\bar{w}'(\bar{\zeta}))\,[(w(\zeta) - 4T\varphi(\zeta))\bar{\Phi}'(\bar{\zeta}) + \bar{\psi}'(\bar{\zeta})],$$

$$P^{12} = 2[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta}) - 4T\Phi(\zeta)\bar{\Phi}(\bar{\zeta})]. \tag{30}$$

The polar coordinates r and θ in the plane of the circle correspond, by virtue of the mapping, to orthogonal curvilinear coordinates in the physical plane. The stress components P_{rr}, $P_{r\theta}$, and $P_{\theta\theta}$ in these coordinates are related to the complex components by the formulas [1, 2]

$$P_{rr} - P_{\theta\theta} + 2iP_{r\theta} = (\bar{\zeta}/\zeta)(\bar{w}'(\bar{\zeta})/w'(\zeta))P^{11}, \qquad P_{rr} + P_{\theta\theta} = P^{12}. \tag{31}$$

As regards the boundary condition in (27) (where we can set $C = 0$), it becomes the boundary condition for the unit circle:

$$\varphi(\zeta) + (w(\zeta) - 4T\varphi(\zeta))\bar{\varphi}'(\bar{\zeta})/\bar{w}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) = g^1, \qquad \zeta \in \gamma. \tag{32}$$

Solution of this problem gives the complex stresses (30) and thereby the real stresses (2) or (31).

Using the small-parameter method, we construct an approximate solution of the nonlinear boundary-value problem (32). Writing the factor $4T$ as $4T = \alpha/P_0$ and assuming the parameter α to be small (compared to unity), we seek the complex potentials in the form of series in this parameter:

$$\varphi(\zeta) = \sum_{\nu=0}^{\infty} \alpha^\nu \varphi_\nu(\zeta), \qquad \psi(\zeta) = \sum_{\nu=0}^{\infty} \alpha^\nu \psi_\nu(\zeta) \tag{33}$$

(φ_ν and ψ_ν are the component potentials). By virtue of (30) and (31), similar expansions are written for the stresses:

$$\begin{gathered} P^{11} = \sum_{\nu=0}^{\infty} \alpha^\nu P^{11}_\nu, \qquad P^{22} = \sum_{\nu=0}^{\infty} \alpha^\nu P^{22}_\nu, \qquad P^{12} = \sum_{\nu=0}^{\infty} \alpha^\nu P^{12}_\nu, \\ P_{rr} = \sum_{\nu=0}^{\infty} \alpha^\nu P^\nu_{rr}, \qquad P_{r\theta} = \sum_{\nu=0}^{\infty} \alpha^\nu P^\nu_{r\theta}, \qquad P_{\theta\theta} = \sum_{\nu=0}^{\infty} \alpha^\nu P^\nu_{\theta\theta}. \end{gathered} \tag{34}$$

Here the stress components are expressed in terms of the component potentials:

$$P^{11}_\nu = \bar{P}^{22}_\nu = -(2/\bar{w}')[w\bar{\Phi}'_\nu + \bar{\psi}'_\nu - P_0^{-1}(\varphi\bar{\Phi}')_{\nu-1}], \qquad P^{12}_\nu = 2[\Phi_\nu + \bar{\Phi}_\nu - P_0^{-1}(\Phi\bar{\Phi})_{\nu-1}],$$

$$\Phi_\nu = \frac{\varphi'_\nu}{w'}, \qquad (\varphi\bar{\Phi}')_{\nu-1} = \sum_{k=0}^{\nu-1} \varphi_k \bar{\Phi}'_{\nu-1-k}, \qquad (\Phi\bar{\Phi})_{\nu-1} = \sum_{k=0}^{\nu-1} \Phi_k \bar{\Phi}_{\nu-1-k}, \tag{35}$$

$$P^\nu_{rr} + P^\nu_{\theta\theta} = P^{12}_\nu, \qquad P^\nu_{rr} - P^\nu_{\theta\theta} + 2iP^\nu_{r\theta} = (\bar{\zeta}/\zeta)(\bar{w}'/w')P^{11}_\nu.$$

We determine the nth approximation of the stresses (34) by finite sums (the approximation order is determined by the required accuracy of solution):

$$\begin{gathered} P^{11}_{(n)} = \sum_{\nu=0}^{n} \alpha^\nu P^{11}_\nu, \qquad P^{22}_{(n)} = \sum_{\nu=0}^{n} \alpha^\nu P^{22}_\nu, \qquad P^{12}_{(n)} = \sum_{\nu=0}^{n} \alpha^\nu P^{12}_\nu, \\ P^{(n)}_{rr} = \sum_{\nu=0}^{n} \alpha^\nu P^\nu_{rr}, \qquad P^{(n)}_{r\theta} = \sum_{\nu=0}^{n} \alpha^\nu P^\nu_{r\theta}, \qquad P^{(n)}_{\theta\theta} = \sum_{\nu=0}^{n} \alpha^\nu P^\nu_{\theta\theta}. \end{gathered} \tag{36}$$

To find the potential components, we substitute (33) into condition (32). Comparing the coefficients of the same powers of the parameter on both sides of the equality, we obtain the sequence of linear problems of the same type for the potentials φ_ν and ψ_ν:

$$\begin{gathered} \varphi_\nu + (w/\bar{w}')\bar{\varphi}'_\nu + \bar{\psi}_\nu = g^1\delta_{\nu 0} + (1/(P_0\bar{w}'))(\varphi\bar{\varphi}')_{\nu-1} \quad \text{on} \quad \gamma, \\ (\varphi\bar{\varphi}')_{\nu-1} = \sum_{k=0}^{\nu-1} \varphi_k \bar{\varphi}'_{\nu-1-k} \qquad (\nu = 0, 1, 2, \ldots). \end{gathered} \tag{37}$$

Here the first problem ($\nu = 0$) coincides with the problem of linear elasticity and for each of the next problems, the right side of the boundary condition is calculated from the solutions of the previous problems.

5. We consider the problem of the extension of a plate with a hole within the framework of a nonlinear model. Let an unbounded plate with a hole be in equilibrium under bulk tension produced by stress P_0 at infinity. It is assumed that in a deformed state, the contour of the hole is free of tractions and has the shape of an ellipse with semiaxes a and b ($a > b$). We find the stress field in the plate and study its properties.

Cartesian axes are made coincident with the axes of the ellipse so that the x axis is directed along the major axis of the ellipse. The equation of the ellipse (contour L) is then written as $x^2/a^2 + y^2/b^2 = 1$. The conformal mapping of the exterior of the ellipse in the z plane onto the exterior of a unit circle in the ζ plane is defined by the transformation

$$z = x + iy = w(\zeta) = n(\zeta + m/\zeta), \qquad \zeta = r\exp(i\theta) \in K,$$
$$0 < n = (a+b)/2 < \infty, \qquad 0 < m = (a-b)/(a+b) < 1, \tag{38}$$

where the parameters m and n give, respectively, the shape and dimensions of the hole (as $m \to 0$, the ellipse degenerates into a circle, and as $m \to 1$, it is degenerated into a rectilinear slit). Consequently, the relation between the Cartesian and curvilinear coordinates in the z plane is given by the formulas $x = n(r+m/r)\cos\theta$ and $y = n(r-m/r)\sin\theta$. In this case, the curvilinear coordinates r and θ are elliptic, i.e., the curves $r = \text{const}$ correspond to ellipses and the curves $\theta = \text{const}$ to hyperbolas (the boundary ellipse corresponds to $r = 1$):

$$\frac{x^2}{(r+m/r)^2} + \frac{y^2}{(r-m/r)^2} = n^2, \qquad \frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 4mn^2.$$

The boundary conditions on the hole contour and at infinity are written in real and complex forms:

$$p_x = p_y = 0; \qquad p^1 = p_x + ip_y = 0, \quad g^1 = i\int_0^s p^1\,ds = 0 \quad \text{for} \quad r = 1; \tag{39}$$

$$P_{xx}^{\infty} = P_{yy}^{\infty} = P_0, \quad P_{xy}^{\infty} = 0; \qquad P_{\infty}^{11} = \bar{P}_{\infty}^{22} = 0, \quad P_{\infty}^{12} = 2P_0 \quad \text{for} \quad r = \infty. \tag{40}$$

By virtue of (39), the condition for the potentials (21) (where we can set $C = 0$) becomes homogeneous; after the conformal transformation (38), it takes the form

$$\bar{w}(\bar{\zeta}) - \bar{\sigma}(\bar{\zeta})\chi(\zeta) + \tau(\zeta) = 0, \qquad \chi(\zeta) = \sigma'(\zeta)/w'(\zeta), \qquad \zeta \in \gamma.$$

Using the analytic continuation, we write this condition as the following functional equation for the exterior of the circle K:

$$\bar{w}(1/\zeta) - \bar{\sigma}(1/\zeta)\chi(\zeta) + \tau(\zeta) = 0. \qquad \zeta \in K. \tag{41}$$

We seek a function $\sigma(\zeta)$ that is similar in form to the mapping (38) and contain two free real parameters c and d. The function $\tau(\zeta)$ is then given by Eq. (41). As a result, the potentials take the form

$$\sigma(\zeta) = n\Big(c\zeta + \frac{md}{\zeta}\Big), \qquad \tau(\zeta) = n\Big(\frac{c + m\,d\,\zeta^2}{\zeta}\,\frac{c\zeta^2 - md}{\zeta^2 - m} - \frac{1 + m\zeta^2}{\zeta}\Big). \tag{42}$$

The potentials (42) correspond to the following expressions for the stresses (19) and (20) [transformed by the mapping (38)]:

$$P^{12} = \frac{1}{2T}\Big(1 - \frac{c\zeta^2 - md}{\zeta^2 - m}\,\frac{c\bar{\zeta}^2 - md}{\bar{\zeta}^2 - m}\Big),$$
$$P^{11} = \frac{\bar{\zeta}^2}{2T(\bar{\zeta}^2 - m)}\Big(\frac{2m(d-c)(\zeta\bar{\zeta} - 1)}{(\bar{\zeta}^2 - m)^2}\,\frac{c\zeta - md\bar{\zeta}}{\zeta} + \frac{c - md\bar{\zeta}^2}{\bar{\zeta}^2}\,\frac{c\bar{\zeta}^2 - md}{\bar{\zeta}^2 - m} + \frac{m\bar{\zeta}^2 - 1}{\bar{\zeta}^2}\Big). \tag{43}$$

At infinity, the stresses in the plate have the form $P_{\infty}^{12} = (1 - c^2)/(2T)$ and $P_{\infty}^{11} = (m/(2T))(1 - cd)$. Consequently, the conditions at infinity (40) are satisfied if

$$d = 1/c, \qquad c^2 = 1 - 4TP_0. \tag{44}$$

According to formulas (31), (38), (43), and (44) [where $\zeta = r\exp(i\theta)$ and $\bar{\zeta} = r\exp(-i\theta)$], the elliptic stress components are written as

$$P_{rr} = \frac{P_0}{c^2E^2}\,(r^2 - 1)(r^4 - m^2)(c^2r^2 - m^2), \quad P_{r\theta} = \frac{P_0}{c^2E^2}\,2mr^2(r^2 - 1)(c^2r^2 - m^2)\sin 2\theta,$$
$$P_{\theta\theta} = \frac{P_0}{c^2E^2}\,[2(c^2r^4 - m^2)E - (r^2 - 1)(r^4 - m^2)(c^2r^2 - m^2)], \tag{45}$$
$$E = r^4 + m^2 - 2mr^2\cos 2\theta.$$

Formulas (45) gives an exact solution of the problem. Thus, the stress distribution in a plate with a hole depends on the tensile force, the elastic constant of the material, and the shape of the hole and does not dependent of the hole dimensions. It follows from the expression for c^2 (44) that the solution obtained is valid for the applied force only when $P_0 < 1/(4T)$. On the hole contour L $(r = 1)$, the stresses are

$$P_{rr}^L = P_{r\theta}^L = 0, \qquad P_{\theta\theta}^L = \frac{2P_0}{c^2}\frac{c^2 - m^2}{1 + m^2 - 2m\cos 2\theta}. \tag{46}$$

For definiteness, we assume that $P_0 > 0$. With allowance for the definition of the parameters and the limitation on the tension, we obtain $0 < P_0 < 1/(4T)$, $0 < c^2 < 1$, and $0 < m^2 < 1$. By virtue of the inequality $1 + m^2 - 2m\cos 2\theta \geqslant (1 - m)^2 > 0$, from (46) it follows that the sign of the stress $P_{\theta\theta}^L$ is determined by the sign of the difference $c^2 - m^2$. We consider different cases.

In the case of weak tension, where $P_0 < (1 - m^2)/(4T)$ $(c^2 > m^2)$, we have $P_{\theta\theta}^L > 0$, i.e., the hole contour is extended and

$$P_+ = (P_{\theta\theta}^L)_{\max} = \frac{2P_0}{c^2}\frac{c^2 - m^2}{(1 - m)^2} \quad \text{for} \quad 2\theta = 0 and 2\pi,$$

$$P_- = (P_{\theta\theta}^L)_{\min} = \frac{2P_0}{c^2}\frac{c^2 - m^2}{(1 + m)^2} \quad \text{for} \quad 2\theta = \pi and 3\pi,$$

where the values $2\theta = 0$ and 2π correspond to points on the major axis of the ellipse and the values $2\theta = \pi$ and 3π correspond to those on the minor axis. When $c^2 > m^2$ and $0 < \alpha = 4TP_0 < 1$, we can set $c^2 = 1 - \alpha$ and $m^2 = 1 - \sqrt{\alpha}$. Consequently,

$$c^2 = m^2(2 - m^2), \qquad P_+ = \frac{4P_0}{2 - m^2}\frac{1 + m}{1 - m}, \qquad P_- = \frac{4P_0}{2 - m^2}\frac{1 - m}{1 + m}.$$

It follows that in the limiting case of a circular hole, $P_+ \to 2P_0$ and $P_- \to 2P_0$ as $m \to 0$, i.e., on the boundary circle, the tensile stress is constant and twice the applied stress. In the limiting case of a rectilinear slit, we find that $P_+ \to \infty$ and $P_- \to 0$ as $m \to 1$, i.e., the tensile stress tends to infinity at the slit ends and is equal to zero in the middle of its sides.

In the case of neutral tension, where $P_0 = (1 - m^2)/(4T)$ $(c^2 = m^2)$, we have $P_{\theta\theta}^L = 0$, i.e., the hole contour is neutral. Thus, there exists a tensile force (which depends on the properties of the material and the shape of the hole) for which the hole contour is neither compressed nor extended. In the limiting cases, we obtain $P_0 \to 1/(4T)$ as $m \to 0$ and $P_0 \to 0$ as $m \to 1$, i.e., for a circular contour, this force coincides with the maximum tension, and for a slit, it coincides with the minimum tension.

Finally, for strong tension, where $P_0 > (1 - m^2)/(4T)$ $(c^2 < m^2)$, we have $P_{\theta\theta}^L < 0$, i.e., the hole contour is compressed, so that

$$P_- = -\frac{2P_0}{c^2}\frac{m^2 - c^2}{(1 - m)^2} \quad \text{for} \quad 2\theta = 0,\ 2\pi, \qquad P_+ = -\frac{2P_0}{c^2}\frac{m^2 - c^2}{(1 + m)^2} \quad \text{for} \quad 2\theta = \pi,\ 3\pi.$$

When $c^2 < m^2$ and $0 < \alpha = 4TP_0 < 1$, we can set $c^2 = 1 - \alpha$ and $m^2 = 1 - \alpha^2$. Then,

$$c^2 = 1 - \sqrt{1 - m^2}, \qquad P_- = -\frac{2P_0}{(1 - m)^2}\sqrt{1 - m^2}, \qquad P_+ = -\frac{2P_0}{(1 + m)^2}\sqrt{1 - m^2}.$$

It is clear that in the limiting case of a circular hole, $P_- \to -2P_0$ and $P_+ \to -2P_0$ as $m \to 0$, i.e., on a circle, the tensile force is constant and twice the applied force. In the case of a slit, we have $P_- \to -\infty$ and $P_+ \to 0$ as $m \to 1$, i.e., the compressive force is infinitely large at the ends of the slit and is equal to zero in the middle of the sides.

It also follows from (45) that for $r^* = m/c$, the stresses can be written in a form similar to the contour stresses (46):

$$P_{rr}^* = P_{r\theta}^* = 0, \qquad P_{\theta\theta}^* = -\frac{2P_0(c^2 - m^2)}{c^4 + m^2 - 2mc^2\cos 2\theta},$$

$$c^4 + m^2 - 2mc^2\cos 2\theta \geqslant (c^2 - m)^2 > 0.$$

Thus, an elliptic contour L^* exists such that it is confocal with the boundary ellipse L and, like the latter, undergoes only tension–compression. The stresses $P^*_{\theta\theta}$ and $P^L_{\theta\theta}$ are opposite in sign. The contour L^* occurs in the plate only in the case of strong tension, where it undergoes tension and encompasses the compressed contour L: $c^2 < m^2$ and $r^* > 1$ (for neutral tension, it coincides with the hole contour, and for weak tension, it lies inside the hole).

6. Using the small-parameter method, we now seek an approximate solution for the above problem in the case where the applied load corresponds to the quantity $\alpha = 4TP_0$ that is small compared to unity.

For the zeroth-order potential components, the boundary-value problem (37) (for $\nu = 0$) coincides with the problem of linear elasticity

$$\varphi_0(\zeta) + (\zeta^2 + m)/(\zeta(1 - m\zeta^2))\bar{\varphi}_0'(\bar{\zeta}) + \bar{\psi}_0(\bar{\zeta}) = 0, \qquad \zeta \in \gamma.$$

Its solution and the corresponding stresses are well known [1]:

$$\varphi_0(\zeta) = (nP_0/2)(\zeta - m/\zeta), \qquad \psi_0(\zeta) = -nP_0(1 + m^2)\zeta/(\zeta^2 - m),$$

$$P^0_{rr} = (P_0/E^2)(r^2 - 1)(r^4 - m^2)(r^2 - m^2),$$

$$P^0_{r\theta} = (P_0/E^2)2mr^2(r^2 - 1)(r^2 - m^2)\sin 2\theta, \tag{47}$$

$$P^0_{\theta\theta} = (P_0/E^2)(r^4 - m^2)[2E - (r^2 - 1)(r^2 - m^2)].$$

These results can also be obtained from the nonlinear formulas (26), (42), and (45) for $c \approx 1$ and they correspond to weak tension.

The first-order potential components are determined from the problem (37) ($\nu = 1$):

$$\varphi_1(\zeta) + \frac{\zeta^2 + m}{\zeta(1 - m\zeta^2)}\bar{\varphi}_1'(\bar{\zeta}) + \bar{\psi}_1(\bar{\zeta}) = \frac{nP_0}{4}\frac{\zeta^2 - m}{\zeta}\frac{1 + m\zeta^2}{1 - m\zeta^2}, \qquad \zeta \in \gamma.$$

These potentials and the corresponding stresses are determined using the functional-equation method [8]:

$$\varphi_1(\zeta) = (nP_0/8)(\zeta - 3m/\zeta), \qquad \psi_1(\zeta) = -nP_0m^2\zeta/(\zeta^2 - m),$$

$$P^1_{rr} = -(P_0/E^2)m^2(r^2 - 1)(r^4 - m^2), \qquad P^1_{r\theta} = -(P_0/E^2)2m^3r^2(r^2 - 1)\sin 2\theta, \tag{48}$$

$$P^1_{\theta\theta} = -(P_0/E^2)m^2[2E - (r^2 - 1)(r^4 - m^2)].$$

Finally, the second-order potential components are determined from the problem (37) for $\nu = 2$:

$$\varphi_2(\zeta) + \frac{\zeta^2 + m}{\zeta(1 - m\zeta^2)}\bar{\varphi}_2'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) = \frac{nP_0}{8}\frac{2m(\zeta^4 - 1) + (1 - 3m^2)\zeta^2}{\zeta(1 - m\zeta^2)}, \qquad \zeta \in \gamma.$$

Using the same method, we infer that these potentials and the corresponding stresses have the form

$$\varphi_2(\zeta) = (nP_0/8)(\zeta - 3m/\zeta), \qquad \psi_2(\zeta) = -(nP_0/8)(1 + 9m^2)\zeta/(\zeta^2 - m),$$

$$P^2_{rr} = (P_0/(8E^2))(r^2 - 1)(r^4 - m^2)(r^2 - 9m^2),$$

$$P^2_{r\theta} = (P_0/(8E^2))2mr^2(r^2 - 1)(r^2 - 9m^2)\sin 2\theta, \tag{49}$$

$$P^2_{\theta\theta} = (P_0/(8E^2))[2E(r^4 - 9m^2) - (r^2 - 1)(r^4 - m^2)(r^2 - 9m^2)].$$

Thus, according to (35) and (36), as a zeroth approximation, the stresses are given by formulas (47):

$$P^{(0)}_{rr} = P^0_{rr} = (P_0/E^2)(r^2 - 1)(r^4 - m^2)(r^2 - m^2),$$

$$P^{(0)}_{r\theta} = P^0_{r\theta} = (P_0/E^2)2mr^2(r^2 - 1)(r^2 - m^2)\sin 2\theta, \tag{50}$$

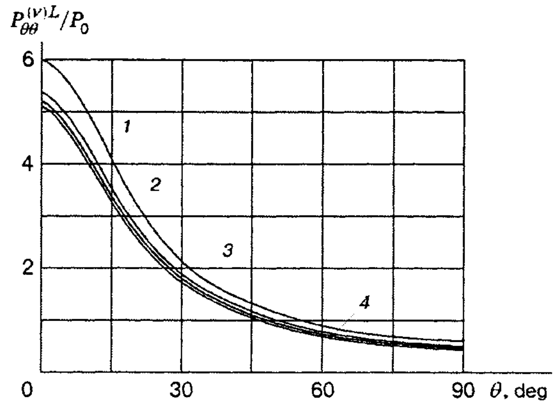

Fig. 1

$$P_{\theta\theta}^{(0)} = P_{\theta\theta}^0 = (P_0/E^2)[2E(r^4 - m^2) - (r^2 - 1)(r^4 - m^2)(r^2 - m^2)];$$

as a first approximation, the stresses are given by formulas (47) and (48):

$$P_{rr}^{(1)} = P_{rr}^0 + \alpha P_{rr}^1 = (P_0/E^2)(r^2 - 1)(r^4 - m^2)[r^2 - m^2(1 + \alpha)],$$
$$P_{r\theta}^{(1)} = P_{r\theta}^0 + \alpha P_{r\theta}^1 = (P_0/E^2)2mr^2(r^2 - 1)[r^2 - m^2(1 + \alpha)] \sin 2\theta, \quad (51)$$
$$P_{\theta\theta}^{(1)} = P_{\theta\theta}^0 + \alpha P_{\theta\theta}^1 = (P_0/E^2)\{2E[r^4 - m^2(1 + \alpha)] - (r^2 - 1)(r^4 - m^2)[r^2 - m^2(1 + \alpha)]\};$$

formulas (47)–(49) give the stresses as a second approximation:

$$P_{rr}^{(2)} = P_{rr}^0 + \alpha P_{rr}^1 + \alpha^2 P_{rr}^2 = (P_0/E^2)(r^2 - 1)(r^4 - m^2)[r^2(1 + \alpha^2/8) - m^2(1 + \alpha + (9/8)\alpha^2)],$$
$$P_{r\theta}^{(2)} = P_{r\theta}^0 + \alpha P_{r\theta}^1 + \alpha^2 P_{r\theta}^2 = (P_0/E^2)2mr^2(r^2 - 1)[r^2(1 + \alpha^2/8) - m^2(1 + \alpha + (9/8)\alpha^2)] \sin 2\theta,$$
$$P_{\theta\theta}^{(2)} = P_{\theta\theta}^0 + \alpha P_{\theta\theta}^1 + \alpha^2 P_{\theta\theta}^2 = (P_0/E^2)\{2E[r^4(1 + \alpha^2/8) - m^2(1 + \alpha + (9/8)\alpha^2)] \quad (52)$$
$$- (r^2 - 1)(r^4 - m^2)[r^2(1 + \alpha^2/8) - m^2(1 + \alpha + (9/8)\alpha^2)]\}.$$

On the hole contour ($r = 1$), the nonzero stresses in the approximate solutions (50)–(52) are

$$P_{\theta\theta}^{(0)L} = (2P_0/E_1)(1 - m^2), \qquad P_{\theta\theta}^{(1)L} = (2P_0/E_1)[1 - m^2(1 + \alpha)],$$
$$P_{\theta\theta}^{(2)L} = (2P_0/E_1)[1 + \alpha^2/8 - m^2(1 + \alpha + (9/8)\alpha^2)], \quad (53)$$
$$E_1 = 1 + m^2 - 2m \cos 2\theta \qquad (E_1 \geqslant (1 - m)^2 > 0).$$

Equations (46) and (53) lead to the relations

$$P_{\theta\theta}^{(0)L} - P_{\theta\theta}^{(1)L} = (P_0/E_1)2m\alpha, \qquad P_{\theta\theta}^{(1)L} - P_{\theta\theta}^{(2)L} = (P_0/(4E_1))(9m^2 - 1)\alpha^2,$$
$$P_{\theta\theta}^{(0)L} - P_{\theta\theta}^{L} = (P_0/(c^2E_1))2m^2\alpha, \qquad P_{\theta\theta}^{(1)L} - P_{\theta\theta}^{L} = (P_0/(c^2E_1))2m^2\alpha^2,$$
$$P_{\theta\theta}^{(2)L} - P_{\theta\theta}^{L} = (P_0/(4c^2E_1))\alpha^2(c^2 - m^2 + 9m^2\alpha),$$

which show that for weak tension ($c^2 > m^2$) of the plate with a narrow hole ($m > 1/3$), the approximate stresses at each boundary point decrease with increase in the approximation order, remaining higher than the exact stresses.

The behavior of the contour stresses $P_{\theta\theta}^{(0)L}/P_0$, $P_{\theta\theta}^{(1)L}/P_0$, $P_{\theta\theta}^{(2)L}$, and $P_{\theta\theta}^{L}/P_0$ corresponding to the approximate and exact solutions obtained for $a/b = 3$, $m = 0.5$, $\alpha = 0.3$, and $c^2 = 0.7$ for weak tension is shown in Fig. 1 (curves 1–4). One can see that allowance for nonlinearity leads to a decrease in the contour stresses in comparison to the linear solution. This decrease depends on the applied load, the properties of the material, and the shape of the hole; it is maximum in the region of maximum stresses (at points of large curvature of the contour) and it is minimum in the region of minimum stresses (at points of small curvature of the contour). In addition, the behavior of the stress curves shows that the approximate solutions converge to the exact solution.

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